

MONETARY VALUE MEASURES IN A CATEGORY OF PROBABILITY SPACES

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In this paper, we generalize the notion of monetary value measures developed in [Adachi, 2014] by extending their base category from the category χ to the category **Prob** introduced in [Adachi and Ryu, 2016].

For those who are not familiar with financial risk management and/or monetary value measures, please refer to Section 2 of [Adachi, 2014].

1. A CATEGORY OF PROBABILITY SPACES

In this section we overview a theory of a category of probability spaces. Please refer to [Adachi and Ryu, 2016] for the full discussions and proofs about the contents of this section.

Let $\bar{X} := (X, \Sigma_X, \mathbb{P}_X)$, $\bar{Y} := (Y, \Sigma_Y, \mathbb{P}_Y)$ and $\bar{Z} := (Z, \Sigma_Z, \mathbb{P}_Z)$ be probability spaces.

Definition 1.1. [Category **Prob**] A category **Prob** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{Prob}(\bar{X}, \bar{Y}) := \{f^- \mid f : \bar{Y} \rightarrow \bar{X}\}$$

is a measurable function satisfying $\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$,

where f^- is a symbol corresponding uniquely to a measurable function f .

We fix the state space be a measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for a simplicity. $\mathcal{L}^\infty(\bar{X})$ is a vector space consisting of \mathbb{R} -valued random variables v such that \mathbb{P}_X -ess sup $_{x \in X} |v(x)| < \infty$, while $\mathcal{L}^1(\bar{X})$ is a vector space consisting of \mathbb{R} -valued random variables v such that $\int_X |v| d\mathbb{P}_X$ has a finite value. For two random variables u_1 and u_2 , we write $u_1 \sim_{\mathbb{P}_X} u_2$ when $\mathbb{P}_X(u_1 \neq u_2) = 0$, and write $u_1 \lesssim_{\mathbb{P}_X} u_2$ when $\mathbb{P}_X(u_1 > u_2) = 0$. Note that $u_1 \lesssim_{\mathbb{P}_X} u_2$ and $u_2 \lesssim_{\mathbb{P}_X} u_1$ iff $u_1 \sim_{\mathbb{P}_X} u_2$. $L^\infty(\bar{X})$ and $L^1(\bar{X})$ are quotient spaces $\mathcal{L}^\infty(\bar{X}) / \sim_{\mathbb{P}_X}$ and $\mathcal{L}^1(\bar{X}) / \sim_{\mathbb{P}_X}$, respectively.

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Definition 1.2. [Functor \mathbf{L}] A functor $\mathbf{L} : \mathbf{Prob} \rightarrow \mathbf{Set}$ is defined by:

$$\begin{array}{ccccc} X & \bar{X} & \xrightarrow{\mathbf{L}} & \mathbf{L}\bar{X} & := L^\infty(\bar{X}) & \ni [u]_{\sim_{\mathbb{P}_X}} \\ \uparrow f & \downarrow f^- & & \downarrow \mathbf{L}f^- & & \downarrow \mathbf{L}f^- \\ Y & \bar{Y} & \xrightarrow{\mathbf{L}} & \mathbf{L}\bar{Y} & := L^\infty(\bar{Y}) & \ni [u \circ f]_{\sim_{\mathbb{P}_Y}} \end{array}$$

Theorem 1.3. Let f^- be an arrow in $\mathbf{Prob}(\bar{X}, \bar{Y})$. Then, for any $v \in \mathcal{L}^1(\bar{Y})$ there exists a $u \in \mathcal{L}^1(\bar{X})$ such that for every $A \in \Sigma_X$

$$(1.1) \quad \int_A u d\mathbb{P}_X = \int_{f^{-1}(A)} v d\mathbb{P}_Y.$$

Moreover, u is determined uniquely up to \mathbb{P}_X -null sets. In other words, if there are two $u_1, u_2 \in \mathcal{L}^1(\bar{X})$ both satisfying (1.1), then $u_1 \sim_{\mathbb{P}_X} u_2$.

We write a version of this u by $E^{f^-}(v)$, and call it a **conditional expectation of v along f^-** . Therefore,

$$(1.2) \quad \int_A E^{f^-}(v) d\mathbb{P}_X = \int_{f^{-1}(A)} v d\mathbb{P}_Y.$$

Proposition 1.4. Let f^- and g^- be arrows in \mathbf{Prob} like:

$$\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z}.$$

- (1) For $u \in \mathcal{L}^1(\bar{X})$, $E^{Id_{\bar{X}}}(u) \sim_{\mathbb{P}_X} u$.
- (2) For $v_1, v_2 \in \mathcal{L}^1(\bar{Y})$, $v_1 \sim_{\mathbb{P}_Y} v_2$ implies $E^{f^-}(v_1) \sim_{\mathbb{P}_X} E^{f^-}(v_2)$.
- (3) For $w \in \mathcal{L}^1(\bar{Z})$, $E^{f^-}(E^{g^-}(w)) \sim_{\mathbb{P}_X} E^{g^- \circ f^-}(w)$.

Definition 1.5. [Functor \mathcal{E}] A functor $\mathcal{E} : \mathbf{Prob}^{op} \rightarrow \mathbf{Set}$ is defined by:

$$\begin{array}{ccccc} X & \bar{X} & \xrightarrow{\mathcal{E}} & \mathcal{E}\bar{X} & := L^1(\bar{X}) & \ni [E^{f^-}(v)]_{\sim_{\mathbb{P}_X}} \\ \uparrow f & \downarrow f^- & & \uparrow \mathcal{E}f^- & & \uparrow \mathcal{E}f^- \\ Y & \bar{Y} & \xrightarrow{\mathcal{E}} & \mathcal{E}\bar{Y} & := L^1(\bar{Y}) & \ni [v]_{\sim_{\mathbb{P}_Y}} \end{array}$$

We call \mathcal{E} a **conditional expectation functor**.

Proposition 1.6. Let $f^- : \bar{X} \rightarrow \bar{Y}$ be a \mathbf{Prob} -arrow, $u, v \in \mathcal{L}^1(\bar{Y})$ and $\alpha, \beta \in \mathbf{R}$.

- (1) *Linearity:* $E^{f^-}(\alpha u + \beta v) \sim_{\mathbb{P}_X} \alpha E^{f^-}(u) + \beta E^{f^-}(v)$.
- (2) *Positivity:* $E^{f^-}(v) \succeq_{\mathbb{P}_X} 0$ if $v \succeq_{\mathbb{P}_Y} 0$.

Theorem 1.7. Let $f^- : \bar{X} \rightarrow \bar{Y}$ be a \mathbf{Prob} -arrow, $u \in \mathcal{L}^1(\bar{Y})$ and $w \in \mathcal{L}^\infty(\bar{X})$. Then we have

$$(1.3) \quad E^{f^-}((w \circ f) \cdot u) \sim_{\mathbb{P}_X} w \cdot E^{f^-}(u).$$

2. MONETARY VALUE MEASURES

A monetary value measure is defined as a presheaf on **Prob**.

Definition 2.1. [Monetary Value Measures] A *monetary value measure* is a contravariant functor

$$\Phi : \mathbf{Prob}^{op} \rightarrow \mathbf{Set}$$

defined by

$$\begin{array}{ccccc} X & \bar{X} & \xrightarrow{\Phi} & \Phi \bar{X} & := L^1(\bar{X}) \ni [\varphi^{f^-}(v)]_{\sim_{\mathbb{P}_X}} \\ \uparrow f & \downarrow f^- & & \uparrow \Phi f^- & \uparrow \Phi f^- \\ Y & \bar{Y} & \xrightarrow{\Phi} & \Phi \bar{Y} & := L^1(\bar{Y}) \ni [v]_{\sim_{\mathbb{P}_Y}} \end{array}$$

where φ^{f^-} satisfies

- (1) *Cash invariance:* $(\forall v \in \mathcal{L}^\infty(\bar{Y}))(\forall u \in \mathcal{L}^\infty(\bar{X}))$
 $\varphi^{f^-}(v + u \circ f) \sim_{\mathbb{P}_X} \varphi^{f^-}(v) + u,$
- (2) *Monotonicity:* $(\forall v_1 \in \mathcal{L}^\infty(\bar{Y}))(\forall v_2 \in \mathcal{L}^\infty(\bar{Y}))$
 $v_1 \lesssim_{\mathbb{P}_Y} v_2 \Rightarrow \varphi^{f^-}(v_1) \lesssim_{\mathbb{P}_X} \varphi^{f^-}(v_2),$
- (3) *Normalization:* $\varphi^{f^-}(0_Y) \sim_{\mathbb{P}_X} 0_X$ if f^- is measure-preserving,
- (4) $v \in \mathcal{L}^\infty(\bar{Y})$ implies $\varphi^{f^-}(v) \in \mathcal{L}^\infty(\bar{X})$ if f^- is measure-preserving.

We sometimes write $\Phi[\varphi]$ for Φ for explicitly noting that arrows mapped by Φ are determined by φ .

At this point, we do not require the monetary value measures to satisfy familiar conditions such as concavity or positive homogeneity. Instead of doing so, we want to see what kind of properties are deduced from this minimal setting.

The most crucial point of Definition 2.1 is that φ does not move only in the direction of time but also moves over several absolutely continuous probability measures *internally*. This means we have a possibility to develop risk measures including ambiguity within this formulation.

Another key point of Definition 2.1 is that φ is a contravariant functor. So, for any pair of arrows $\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z}$ in **Prob**, we have

$$(2.1) \quad \Phi Id_{\bar{X}} = Id_{L^1(\bar{X})} \text{ and } \Phi f^- \circ \Phi g^- = \Phi(g^- \circ f^-).$$

As an example of monetary value measures, we will introduce a notion of entropic value measures that depend on conditional expectations $E^{f^-}(v)$ of v along f^- .

Before introducing entropic value measures, we need the following lemma.

Lemma 2.2. *Let $f^- : \bar{X} \rightarrow \bar{Y}$ be a measure-preserving arrow in **Prob**. Then, $v \in \mathcal{L}^\infty(\bar{Y})$ implies $E^{f^-}(v) \in \mathcal{L}^\infty(\bar{X})$.*

Proof. Since $v \in \mathcal{L}^\infty(\bar{Y})$, there exists a non-negative $M \geq 0$ such that $-M \lesssim_{\mathbb{P}_Y} v \lesssim_{\mathbb{P}_Y} M$. Then, by Proposition 1.6,

$$0 \lesssim_{\mathbb{P}_X} E^{f^-}(M - v) \sim_{\mathbb{P}_X} E^{f^-}(M) - E^{f^-}(v).$$

On the other hand, we have

$$E^{f^-}(M) \sim_{\mathbb{P}_X} M E^{f^-}(1_Y) \sim_{\mathbb{P}_X} M$$

since f is measure preserving. Therefore, we obtain $E^{f^-}(v) \lesssim_{\mathbb{P}_X} M$. Similarly, we have $-M \lesssim_{\mathbb{P}_X} E^{f^-}(v)$. So we get $E^{f^-}(v) \in \mathcal{L}^\infty(\bar{X})$. \square

Proposition 2.3. *[Entropic Value Measures] Let $f^- : \bar{X} \rightarrow \bar{Y}$ be a **Prob**-arrow, and λ be a positive real number. Define a function $\varphi^{f^-} : L^1(\bar{Y}) \rightarrow L^1(\bar{X})$ by*

$$(2.2) \quad \varphi^{f^-}(v) := \lambda^{-1} \log E^{f^-}(e^{\lambda v}), \quad (\forall v \in L^1(\bar{Y})).$$

*Then, $\Phi := \Phi[\varphi]$ is a monetary value measure. We call this Φ an **entropic value measure**.*

Proof. Let $\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z}$ be arrows in **Prob**. In order to show that Φ becomes a contravariant functor, we need to check three points: $\varphi^{Id_{\bar{Z}}}(w) \sim_{\mathbb{P}_Z} w$, $\varphi^{f^-}(\varphi^{g^-}(w)) \sim_{\mathbb{P}_X} \varphi^{g^- \circ f^-}(w)$, and that $w_1 \sim_{\mathbb{P}_Z} w_2$ implies $\varphi^{g^-}(w_1) \sim_{\mathbb{P}_Y} \varphi^{g^-}(w_2)$ for every $w, w_1, w_2 \in \mathcal{L}^1(\bar{Z})$. But, they are straightforward consequences of Proposition 1.4. So, we forward to check if φ^{f^-} satisfies the four conditions of Definition 2.1.

Firstly, we show that $\varphi^{f^-}(v + u \circ f) \sim_{\mathbb{P}_X} \varphi^{f^-}(v) + u$ for $v \in \mathcal{L}^\infty(\bar{Y})$ and $u \in \mathcal{L}^\infty(\bar{X})$. But by Theorem 1.7, we have

$$\begin{aligned} \varphi^{f^-}(v + u \circ f) &= \lambda^{-1} \log E^{f^-}(e^{\lambda(v+u \circ f)}) \\ &= \lambda^{-1} \log E^{f^-}(e^{\lambda v} \cdot ((e^{\lambda u}) \circ f)) \\ &\sim_{\mathbb{P}_X} \lambda^{-1} \log \left(e^{\lambda u} \cdot E^{f^-}(e^{\lambda v}) \right) \\ &= u + \varphi^{f^-}(v). \end{aligned}$$

Secondly, we show that $v_1 \lesssim_{\mathbb{P}_Y} v_2$ implies $\varphi^{f^-}(v_1) \lesssim_{\mathbb{P}_X} \varphi^{f^-}(v_2)$ for $v_1, v_2 \in \mathcal{L}^\infty(\bar{Y})$. But this comes from Proposition 1.6.

Thirdly, we show that $\varphi^{f^-}(0_Y) \sim_{\mathbb{P}_X} 0_X$ if f^- is measure-preserving. But this is straightforward like the following:

$$\varphi^{f^-}(0_Y) = \lambda^{-1} \log E^{f^-}(e^{\lambda 0_Y}) = \lambda^{-1} \log E^{f^-}(1_Y) \sim_{\mathbb{P}_X} \lambda^{-1} \log 1_X = 0_X.$$

Lastly, we need to show that $v \in \mathcal{L}^\infty(\bar{Y})$ implies $\varphi^{f^-}(v) \in \mathcal{L}^\infty(\bar{X})$ when f^- is measure-preserving. But this comes from Lemma 2.2. \square

Here are some properties of monetary value measures.

Theorem 2.4. *Let $\Phi = \Phi[\varphi] : \mathbf{Prob}^{op} \rightarrow \mathbf{Set}$ be a monetary value measure, and $\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z}$ be arrows in **Prob**.*

- (1) If f^- is measure-preserving, we have $\Phi f^- \circ Lf^- = Id_{L\bar{X}}$.
- (2) Idempotence: If f^- is measure-preserving, we have $\Phi f^- \circ Lf^- \circ \Phi f^- = \Phi f^-$.
- (3) Local property: $(\forall v_1 \in \mathcal{L}^\infty(\bar{Y}))(\forall v_2 \in \mathcal{L}^\infty(\bar{Y}))(\forall A \in \Sigma_X)$
 $\Phi f^- [1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2]_{\sim_{\mathbb{P}_Y}} = [1_{f^{-1}(A)}]_{\sim_{\mathbb{P}_X}} \Phi f [v_1]_{\sim_{\mathbb{P}_Y}} + [1_{f^{-1}(A^c)}]_{\sim_{\mathbb{P}_X}} \Phi f [v_2]_{\sim_{\mathbb{P}_Y}}.$
- (4) Dynamic programming principle: If g^- is measure-preserving,
 $\varphi^{g^- \circ f^-}(w) = \varphi^{g^- \circ f^-}(\varphi^{g^-}(w) \circ g)$ for $w \in \mathcal{L}^\infty(\bar{Z})$.
- (5) Time consistency: $(\forall w_1 \in \mathcal{L}^\infty(\bar{Z}))(\forall w_2 \in \mathcal{L}^\infty(\bar{Z}))$
 $\varphi^{g^-}(w_1) \lesssim_{\mathbb{P}_Y} \varphi^{g^-}(w_2) \Rightarrow \varphi^{g^- \circ f^-}(w_1) \lesssim_{\mathbb{P}_X} \varphi^{g^- \circ f^-}(w_2).$

Proof. (1) For $u \in \mathcal{L}^\infty(\bar{X})$, $\Phi f^-(Lf^-[u]_{\sim_{\mathbb{P}_X}}) = [\varphi^{f^-}(u \circ f)]_{\sim_{\mathbb{P}_X}}$

But, by cash invariance and normalization, we have $\varphi^{f^-}(u \circ f) = \varphi^{f^-}(0_Y + (u \circ f)) \sim_{\mathbb{P}_X} \varphi^{f^-}(0_Y) + u \sim_{\mathbb{P}_X} 0_X + u = u$.

(2) Immediate by (1).

(3) First, we show that for any $A \in \Sigma_X$ and $v \in \mathcal{L}^\infty(\bar{Y})$,

$$(2.3) \quad 1_A \varphi^{f^-}(v) \sim_{\mathbb{P}_X} 1_A \varphi^{f^-}(1_{f^{-1}(A)}v).$$

Since $v \in \mathcal{L}^\infty(\bar{Y})$, for every $y \in Y$ we have $|v(y)| \lesssim_{\mathbb{P}_Y} \|v\|_{\mathcal{L}^\infty(\bar{Y})}$.
Therefore,

$$1_{f^{-1}(A)}v - 1_{f^{-1}(A^c)}\|v\|_{\mathcal{L}^\infty(\bar{Y})} \lesssim_{\mathbb{P}_Y} 1_{f^{-1}(A)}v + 1_{f^{-1}(A^c)}v \lesssim_{\mathbb{P}_Y} 1_{f^{-1}(A)}v + 1_{f^{-1}(A^c)}\|v\|_{\mathcal{L}^\infty(\bar{Y})}.$$

Then noting that $1_A \circ f = 1_{f^{-1}(A)}$, we have the following sequence of equations by cash invariance and monotonicity.

$$\begin{aligned} \varphi^{f^-}(1_{f^{-1}(A)}v) - \|v\|_{\mathcal{L}^\infty(\bar{Y})}1_{A^c} &\sim_{\mathbb{P}_X} \varphi^{f^-}(1_{f^{-1}(A)}v - (\|v\|_{\mathcal{L}^\infty(\bar{Y})}1_{A^c}) \circ f) \\ &= \varphi^{f^-}(1_{f^{-1}(A)}v - 1_{f^{-1}(A^c)}\|v\|_{\mathcal{L}^\infty(\bar{Y})}) \\ &\lesssim_{\mathbb{P}_X} \varphi^{f^-}(v) \\ &\lesssim_{\mathbb{P}_X} \varphi^{f^-}(1_{f^{-1}(A)}v + 1_{f^{-1}(A^c)}\|v\|_{\mathcal{L}^\infty(\bar{Y})}) \\ &= \varphi^{f^-}(1_{f^{-1}(A)}v + (\|v\|_{\mathcal{L}^\infty(\bar{Y})}1_{A^c}) \circ f) \\ &\sim_{\mathbb{P}_X} \varphi^{f^-}(1_{f^{-1}(A)}v) + \|v\|_{\mathcal{L}^\infty(\bar{Y})}1_{A^c}. \end{aligned}$$

Hence

$$\varphi^{f^-}(1_{f^{-1}(A)}v) - 1_{A^c}\|v\|_{\mathcal{L}^\infty(\bar{Y})} \lesssim_{\mathbb{P}_X} \varphi^{f^-}(v) \lesssim_{\mathbb{P}_X} \varphi^{f^-}(1_{f^{-1}(A)}v) + 1_{A^c}\|v\|_{\mathcal{L}^\infty(\bar{Y})}.$$

By multiplying 1_A , we obtain

$$1_A \varphi^{f^-}(1_{f^{-1}(A)}v) - \lesssim_{\mathbb{P}_X} 1_A \varphi^{f^-}(v) \lesssim_{\mathbb{P}_X} 1_A \varphi^{f^-}(1_{f^{-1}(A)}v).$$

Therefore, we get (2.3).

Next by using (2.3) twice, we have

$$\begin{aligned}
& \varphi^{f^-}(1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2) \\
&= 1_{f^{-1}(A)}\varphi^{f^-}(1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2) + 1_{f^{-1}(A^c)}\varphi^{f^-}(1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2) \\
&\sim_{\mathbb{P}_X} 1_{f^{-1}(A)}\varphi^{f^-}(1_{f^{-1}(A)}(1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2)) + 1_{f^{-1}(A^c)}\varphi^{f^-}(1_{f^{-1}(A^c)}(1_{f^{-1}(A)}v_1 + 1_{f^{-1}(A^c)}v_2)) \\
&= 1_{f^{-1}(A)}\varphi^{f^-}(1_{f^{-1}(A)}v_1) + 1_{f^{-1}(A^c)}\varphi^{f^-}(1_{f^{-1}(A^c)}v_2) \\
&\sim_{\mathbb{P}_X} 1_{f^{-1}(A)}\varphi^{f^-}(v_1) + 1_{f^{-1}(A^c)}\varphi^{f^-}(v_2).
\end{aligned}$$

(4) By (2), we have $\varphi^{g^-}(\varphi^{g^-}(w) \circ g) \sim_{\mathbb{P}_Y} \varphi^{g^-}(w)$ for $w \in \mathcal{L}^\infty(\bar{Z})$.
So by (2.1),

$$\begin{aligned}
\varphi^{g^- \circ f^-}(w) &\sim_{\mathbb{P}_X} \varphi^{f^-}(\varphi^{g^-}(w)) \sim_{\mathbb{P}_X} \varphi^{f^-}(\varphi^{g^-}(\varphi^{g^-}(w) \circ g)) \\
&= (\varphi^{f^-} \circ \varphi^{g^-})(\varphi^{g^-}(w) \circ g) \sim_{\mathbb{P}_X} \varphi^{g^- \circ f^-}(\varphi^{g^-}(w) \circ g).
\end{aligned}$$

(5) Assume $\varphi^{g^-}(w_1) \lesssim_{\mathbb{P}_Y} \varphi^{g^-}(w_2)$. Then, by monotonicity and (2.1),

$$\varphi^{g^- \circ f^-}(w_1) \sim_{\mathbb{P}_X} \varphi^{f^-}(\varphi^{g^-}(w_1)) \lesssim_{\mathbb{P}_X} \varphi^{f^-}(\varphi^{g^-}(w_2)) \sim_{\mathbb{P}_X} \varphi^{g^- \circ f^-}(w_2).$$

□

In Theorem 2.4, two properties, dynamic programming principle and time consistency are usually introduced as axioms ([Detlefsen and Scandolo, 2006]). But, we derive them naturally here from the fact that the monetary value measure is a contravariant functor.

Before ending this section, we mention an interpretation of the Yoneda lemma in our setting.

Theorem 2.5. *[The Yoneda Lemma] For any monetary value measure $\Phi : \mathbf{Prob}^{op} \rightarrow \mathbf{Set}$ and an object \bar{X} in \mathbf{Prob} , there exists a bijective correspondence $y_{\Phi, \bar{X}}$ specified by the following diagram:*

$$\begin{array}{ccc}
y_{\Phi, \bar{X}} : \text{Nat}(\mathbf{Prob}(-, \bar{X}), \Phi) & \xrightarrow{\cong} & L^1(\bar{X}) \\
\alpha & \longmapsto & \alpha_{\bar{X}}(Id_{\bar{X}}^-) \\
\tilde{\mathbf{u}} & \longleftarrow & \mathbf{u}
\end{array}$$

where $\tilde{\mathbf{u}}$ is a natural transformation defined by for any $f^- : \bar{Y} \rightarrow \bar{X}$ in \mathbf{Prob} , $\tilde{\mathbf{u}}_{\bar{Y}}(f^-) := \Phi f^- \mathbf{u}$. Moreover, the correspondence is natural in both Φ and \bar{X} .

It makes sense to consider the representable functor $\mathbf{Prob}(-, \bar{X})$ as a generalized *time domain* with time horizon \bar{X} . Then a natural transformation from $\mathbf{Prob}(-, \bar{X})$ to Φ can be seen as a *stochastic process* that is (in a sense) adapted to Φ , and its corresponding Σ_X -measurable random variable represents a terminal value (payoff) at the horizon.

The Yoneda lemma says that we have a bijective correspondence between those stochastic processes and random variables.

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